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Optimal Maintenance Policies Under Changing Technology and Environment

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SUMMARY & CONCLUSIONS

A new model and solution methods are introduced to find the optimal time of preventive replacement of a working object. Most models consider only the replacement costs and the distribution of the time to failure. Very few models take the random termination time of ongoing projects into account, when project interruption results in additional cost. Traditionally, the object is replaced upon failure. Otherwise it is replaced at the earlier (replace first) or later (replace last) occurrence of either the termination of the project or the scheduled replacement time. The long-term optimum models were formulated in [1] and the solution methodology was illustrated with examples.

This paper presents the single-cycle counterpart of these models which are extended to include repair costs of repairable failures, generated revenues and salvage value of the unit in a one-cycle cost per unit time minimization model. The models are further extended to the case when the unit repeats identical projects in time and the terminating time of the ongoing project or that of a given number of projects is considered in the replace next model, which is also formulated. Conditions are given for the existence of finite optima, and examples illustrate the models and solutions.

A comparison of the alternative models discussed in this paper has an interesting conclusion. Consider the derivatives $h_1(T)$, $h_2(T)$ and $h_3(T)$ of the objective functions of the classical, "replace first', "replace last" models, respectively. They all have an identical first term, which is constant if all distributions are exponential and strictly increasing if at least one distribution is Weibull, which is the case if the irreparable failure is Weibull. In addition to strictly increasing, it is zero at $T = 0$ and tends to infinity if $T \to \infty$. The second term of $h_2(t)$ is always larger than the second term of $h_3(t)$, so the solution of $h_2(t) = 0$ is larger than the solution of $h_3(t) = 0$. The second term of $h_1(t)$ is between those of $h_2(t)$ and $h_3(t)$, so the solution of the first order condition of the classical model is between the solutions of the other two first order conditions. This has a sense. If "replacement last" model is used, then the replacement is often done after time T, when failure occurs or working cycle terminates, so the actual replacement is often performed after time T. However, in the case of "replacement first" the actual replacement often occurs before time T, when either working cycle terminates or unit fails. That is, in using "replacement last" policy the value of T underestimates the actual replacement time, while in the case of "replacement first" the value of T overestimates it. If the classical model is used, then replacement is done at time T unless failure occurs, no project termination time is considered.

The classical model, "replacement first" and "replacement last" models have a simple closed form objective function, so they can be solved by simple computer methods. The most frequently used procedures are discussed in many textbooks on computer methods, for example in [2]. In the general case the "replacement next" model needs simulation to find its optimum, closed form representation of its objective function is only possible in special cases.

1 INTRODUCTION

The uninterrupted working condition of an object or a whole system can be guaranteed by appropriate maintenance and replacement planning. If non-repairable failure occurs, then failure replacement is performed, which is usually more expensive than doing preventive replacement. Before replacements, repairable failures might occur, revenue is generated, and the replaced items still have certain salvage values. These factors also have to be included in any optimum model.

There are two major types of models discussed in literature. If the working cycle repeats until infinity without any change, then the renewal theory is used to formulate the objective function, which is the long-term cost per unit time. If technology and the environment change, then a short-term optimum model is constructed minimizing the cost per unit time in a single cycle. These two types of models are used very often in practice and computer software packages are available for their implementation.

There are several surveys discussing the most frequently applied model variants [3][4][5][6][7][8][9], and the most appropriate model selection depends on the actual circumstances the decision maker is faced with.

In most models there is no attention given to ongoing projects the system is involved in and project interruptions for performing maintenance or replacements. Recently [1] introduced two long-term models. In the "replacement first" policy the unit is replaced before failure either at the end of a random working cycle or at the scheduled preventive replacement time, whichever comes first. If the "replacement last" policy is applied, then replacement is performed either at the end of a random working cycle or at the scheduled preventive replacement time, whichever comes last. The mathematical models were formulated and solution algorithms were suggested to find the optimal decisions.

In this paper the one-cycle variants of the models of [1] are constructed with slight extensions including repair costs of repairable failures, revenue generation as well as the salvage values of the unit upon replacement. In the next section the classical model will be examined with extensions. Section 3 discusses the replacement first model, and the replacement last model will be the subject of Section 4. If the unit performs projects repeatedly and replacement is done when the ongoing project terminates just after the scheduled replacement time, then the above models have to be adjusted accordingly resulting in "replacement next" concepts. That is the subject of Section 5.

2 THE CLASSICAL MODEL

Let $\tilde{X}$ denote the time to failure as a Weibull variable with CDF $F(x)$, pdf $f(x)$ reliability function $\bar{F}(x)$ and failure rate $\rho(x)$. The preventive and failure replacement costs are c_p and c_f, where we assume that $c_f > c_p$. It is also assumed that there are K types of repairable failures with failure rates $\bar{\rho}_k(t)$ and expected number of failures $\bar{M}_k(t)$ in interval $[0,t]$. The revenue generated by the unit in a single working cycle is V, and the salvage value of the unit at time t is $S(t)$. The cost per unit time in any interval $[0,t]$ is given as

$$\frac{c_p + \sum_{k=1}^{K} c_{rk}\bar{M}_k(t) - S(t) - eVt}{t} \quad (1)$$

if preventive replacement is performed at time t, where c_{rk} is the cost of repairing failure of type k and eV is the expected utility generated by the unit in a unit time period $e = E(1/\tilde{Y})$ with $\tilde{Y}$ being the random length of a project. If failure replacement is done, then the term c_p in the numerator is replaced with c_f. If preventive replacement is scheduled at time T, then the expected cost per unit time in a cycle can be given as

$$H_1(T) = \frac{c_p + Q(T)}{T}\bar{F}(T) + \int_0^T \frac{c_f + Q(t)}{t} f(t)dt \quad (2)$$

which is then minimized, where:

$$Q(t) = \sum_{k=1}^{K} c_{rk}\bar{M}_k(t) - S(t) - eVt \quad (3)$$

collects all costs excluding replacement costs. Assuming Weibull failure rates with parameters $\bar{\eta}_k$ and $\bar{\beta}_k$ for failure type k, then the derivative of $H_1(T)$ has the same sign as

$$h_1(T) = \left[(c_f - c_p)\rho(T) + \sum_{k=1}^{K} c_{rk}(\bar{\beta}_k - 1)\frac{T^{\bar{\beta}_k - 1}}{\bar{\eta}_k^{\bar{\beta}_k}}\right] - \left[\frac{c_p + TS'(T) - S(T)}{T}\right] \quad (4)$$

It is a natural assumption that $S'(T)$ is bounded, $S(\infty) = 0$ and $S(0) < c_p$, so $h_1(0) = -\infty$ and $h_1(\infty) = \infty$. Therefore there is always a nonzero, finite optimum, which can be obtained by standard methods.

3 THE "REPLACE FIRST" APPROACH

In formulating the objective function we have to consider the random time to failure $\tilde{X}$, the random length of the project $\tilde{Y}$, and the scheduled preventive maintenance time T, which have 6 different permutations:

1. $\tilde{Y} < T < \tilde{X}$
2. $\tilde{Y} < \tilde{X} < T$
3. $\tilde{X} < \tilde{Y} < T$
4. $\tilde{X} < T < \tilde{Y}$
5. $T < \tilde{X} < \tilde{Y}$
6. $T < \tilde{Y} < \tilde{X}$

The six permutation cases are illustrated in *Figure 1*.

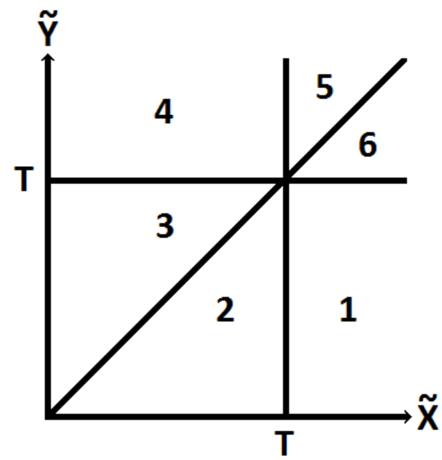

Figure 1. The possible six permutations of $\tilde{X}, \tilde{Y}, T$

In cases 1 and 2 the cycle length is $\tilde{Y}$, in cases 3 and 4 it is $\tilde{X}$ and in the last two cases the cycle length is T. Therefore the expected cost per unit time has three major terms

$$H_2(T) = \int_0^T \int_y^\infty \frac{c_p + Q(y)}{y} f(x)g(y)dxdy + \int_T^\infty \int_T^\infty \frac{c_p + A + Q(T)}{T} f(x)g(y)dxdy + \int_0^T \int_x^\infty \frac{c_f + A + Q(x)}{x} f(x)g(y)dydx$$

$$= \int_0^T \frac{c_p + Q(y)}{y}\bar{F}(y)g(y)dy +$$

$$\frac{c_p + A + Q(T)}{T}\bar{F}(T)\bar{G}(T) + \int_0^T \frac{c_f + A + Q(x)}{x} f(x)\bar{G}(x)dx \quad (5)$$

where *g(t)* and $\bar{G}(t)$ are the pdf and reliability function of $\tilde{Y}$, furthermore A is the project interruption cost.

The derivative of $H_2(T)$ has the same sign as

$$h_2(T) = \left[(c_f - c_p)\rho(T) + \sum_{k=1}^{K} c_{rk}(\bar{\beta}_k - 1)\frac{T^{\bar{\beta}_k - 1}}{\bar{\eta}_k^{\bar{\beta}_k}}\right] - \left[\frac{c_p + A + TS'(T) - S(T)}{T} + A\bar{\rho}(T)\right] \quad (6)$$

by assuming Weibull failure rates for the repairable failures and $\bar{\rho}(T)$ is the "failure rate" $(g(t)/\bar{G}(t))$ of the length of the project. If *S'(T)* is bounded, $S(0) < c_p$ and $S(\infty) = 0$, then $h_2(0) = -\infty$ and if in addition $\lim_{T\to\infty} h_2(T) > 0$, then there is a finite nonzero optimum.

4 THE "REPLACE LAST" APPROACH

Similarly to the previous case we have the six permutations of $\tilde{X}$, $\tilde{Y}$, T. In case 1 the cycle length is T, in cases 2, 3, 4, and 5 the cycle length is $\tilde{X}$, and in case 6 it is $\tilde{Y}$. So the expected cost per unit time in a cycle can be given as follows:

$$\begin{aligned} H_3(T) = &\int_T^\infty \int_y^\infty \frac{c_p + Q(y)}{y} f(x)g(y)dxdy + \\ &\int_0^T \int_T^\infty \frac{c_p + A + Q(T)}{T} f(x)g(y)dxdy + \\ &\int_0^\infty \int_x^\infty \frac{c_f + A + Q(x)}{x} f(x)g(y)dydx + \\ &\int_0^T \int_0^x \frac{c_f + A + Q(x)}{x} f(x)g(y)dydx \\ = &\int_T^\infty \frac{c_p + Q(y)}{y}\bar{F}(y)g(y)dy + \\ &\frac{c_p + A + Q(T)}{T}\bar{F}(T)G(T) + \\ &\int_0^\infty \frac{c_f + A + Q(x)}{x} f(x)\bar{G}(x)dx + \\ &\int_0^T \frac{c_f + A + Q(x)}{x} f(x)G(x)dx \end{aligned} \quad (7)$$

Simple calculation shows that $H_3'(T)$ has the same sign as

$$h_3(T) = \left[(c_f - c_p)\rho(T) + \sum_{k=1}^{K} c_{rk}(\bar{\beta}_k - 1)\frac{T^{\bar{\beta}_k - 1}}{\bar{\eta}_k^{\bar{\beta}_k}}\right] - \left[\frac{c_p + A + TS'(T) - S(T)}{T} - A\frac{g(T)}{G(T)}\right] \quad (8)$$

where *G(t)* is the CDF of $\tilde{Y}$. If it is exponential or Weibull, then $(g(t)/G(t))$ converges to zero as $T \to \infty$ and to infinity as $T \to 0$. Thus $h_3(T)$ converges to infinity as $T \to \infty$, so there is always a finite optimum which is positive if the second term of (8) has a positive limit at zero.

5 THE "REPLACE NEXT" APPROACH

In this case we assume that the unit performs identical projects repeatedly. If it breaks down, then it is replaced immediately, otherwise we can wait until the scheduled preventive replacement time arrives and the unit is replaced when the ongoing project terminates. This termination time depends on T in contrary to the distribution of $\tilde{Y}$ as in the previous two cases. In this case, however, let $\tilde{Z}_{i+1}$ denote the termination time of the ongoing project, then its CDF is given as

$$G_T(t) = \sum_{i=0}^{\infty} P\left(\tilde{Y}_1 + \cdots \tilde{Y}_i < T < \tilde{Y}_1 + \cdots + \tilde{Y}_i + \tilde{Y}_{i+1} < t\right) \quad (9)$$

where for $i = 0$, $\tilde{Y}_1 + \cdots + \tilde{Y}_i = 0$. The pdf of $\tilde{Z}_i$ is the i-fold convolution of *g(t)* which can be denoted by $g_i(t)$, so

$$g_i(t) = \int_0^\infty g_{i-1}(\tau)g(t-\tau)d\tau \quad (10)$$

can be obtained recursively by numerical integration.

The defining domain of $G_T(t)$ is shown in *Figure 2*,

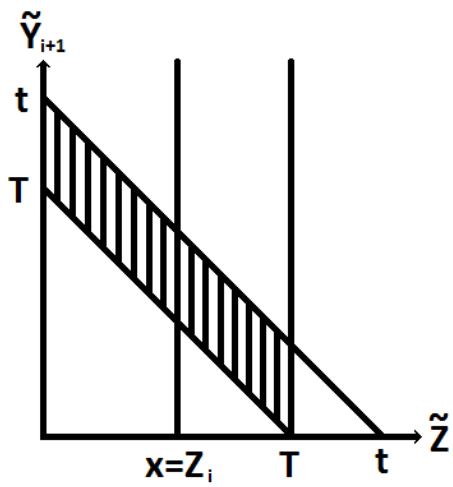

Figure 2. Domain defining $G_T(t)$

and since $g_i(t)$ and $g(t)$ are independent, for $(t \geq T)$

$$\begin{aligned} G_T(t) &= G(t) - G(T) + \sum_{i=1}^{\infty} \int_0^T \int_{T-x}^{t-x} g_i(x)g(y)dydx \\ &= G(t) - G(T) + \sum_{i=1}^{\infty} \int_0^T g_i(x)[G(t-x) - G(T-x)]dx \\ &= G(t) - G(T) + \int_0^T H(x)[G(t-x) - G(T-x)]dx \end{aligned} \quad (11)$$

where

$$H(x) = \sum_{i=1}^{\infty} g_i(x). \quad (12)$$

The "replacement last" model can be applied with *G(t)* being replaced by $G_T(t)$ and in the objective function only cases 4, 5, and 6 are feasible. The objective function becomes

$$H_4(T) = \int_0^T \frac{c_f + Q(x) + A}{x} f(x)dx\bar{G}_T(T) + \int_T^\infty \frac{c_f + Q(x) + A}{x} f(x)\bar{G}_T(x)dx + \int_T^\infty \frac{c_p + Q(y)}{y} \bar{F}(y)g_T(y)dy \quad (13)$$

Since $\bar{G}_T(x)$ and $g_T(y)$ depend on *T*, no simple analytic solution can be obtained in general.

We can also consider a simplified version of this model, when $\tilde{Y}$ is replaced by the termination time of a given number, *N*, of consecutive projects. Then *G(t)* has to be replaced by the CDF of $g_N(t)$, and the two models can be applied without further changes.

6 EXAMPLE

Consider an object with a Weibull non-repairable failure and one type of exponential repairable failure with parameters $\beta = 2, \eta = 1$, and $\bar{\lambda} = 1$. The length of the project is also exponential with $\lambda = 2$. The utility generation is $eV = 200$ per unit time and the salvage value of the object after *t* time periods is $S(t) = 40e^{-t}$. The failure and preventive replacement costs are $c_f = 200$ and $c_p = 50$, respectively, and furthermore the repair cost at each repairable failure is $c_{r1} = 20$. The project interruption cost is $A = 5$.

First we illustrate the replace last and replace next models. In this case:

$$f(t) = 2te^{-t^2},\ F(t) = 1 - e^{-t^2},\ \bar{F}(t) = e^{-t^2}$$
$$\rho(t) = 2t$$
$$\bar{M}_1(t) = t$$
$$Q(t) = 20t - 40e^{-t} - 200t = -180t - 40e^{-t}$$
$$g(t) = 2e^{-2t},\ G(t) = 1 - e^{-2t},\ \bar{G}(t) = e^{-2t}.$$

Figure 3 shows the shape of the objective function of $H_3(T)$ of the replace last model, where the optimal solution is $t^* = 0.20$.

In examining the replace next model, notice first that if $\tilde{Y}_i$ is exponential with $\lambda = 2$, then $\tilde{Y}_1 + \cdots + \tilde{Y}_i$ is gamma with $\alpha = i$ and $\lambda = 2$ with pdf:

$$g_i(t) = \frac{\lambda^i}{\Gamma(i)} t^{i-1} e^{-\lambda t}$$

so

$$G_T(t) = \left(1 - e^{-\lambda t}\right) - \left(1 - e^{-\lambda T}\right) + \int_0^T \lambda\left[1 - e^{-\lambda(t-x)} - 1 + e^{-\lambda(T-x)}\right]dx$$

$$= 1 - e^{-\lambda(t-T)}$$

since

$$\sum_{i=1}^{\infty} g_i(t) = \lambda e^{-\lambda t} \sum_{i=1}^{\infty} \frac{(\lambda t)^{i-1}}{(i-1)!} = \lambda$$

and so in (13),

$$g_T(t) = 2e^{-2(t-T)} \text{ and } \bar{G}_T(t) = e^{-2(t-T)}$$

Figure 4 illustrates $H_4(T)$ and the optimal solution is $t^* = 0.07$.

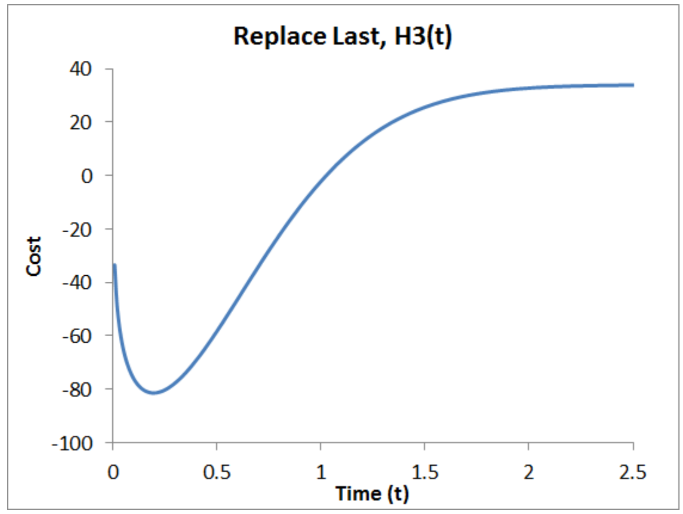

Figure 3.Replace last cost function $H_3(t)$

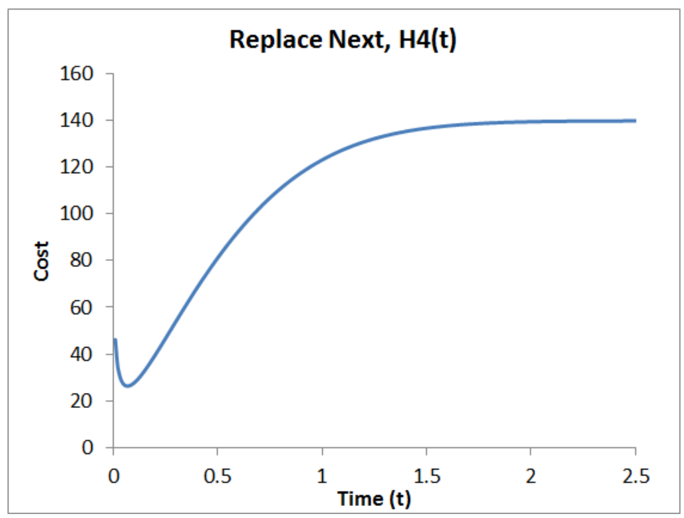

Figure 4. Replace next cost function $H_4(t)$

We also notice that with exponential repairable failures the replace first model cannot be applied, since the first integral term of (5) becomes infinity. In comparing replacement first and last models we now assume that the length of a project is also Weibull with parameters $\bar{\beta} = 2, \bar{\eta} = 2$, keeping all other distributions and data unchanged. In this case

$$g(t) = \frac{2t}{4} e^{-\frac{t^2}{4}},\ G(t) = 1 - e^{-\frac{t^2}{4}} \text{ and } \bar{G}(t) = e^{-\frac{t^2}{4}}.$$

Figures 5 & 6 show the shapes of $H_2(T)$ and $H_3(T)$ and the optimal solutions are $t_2^* = 0.23$ and $t_3^* = 0.13$.

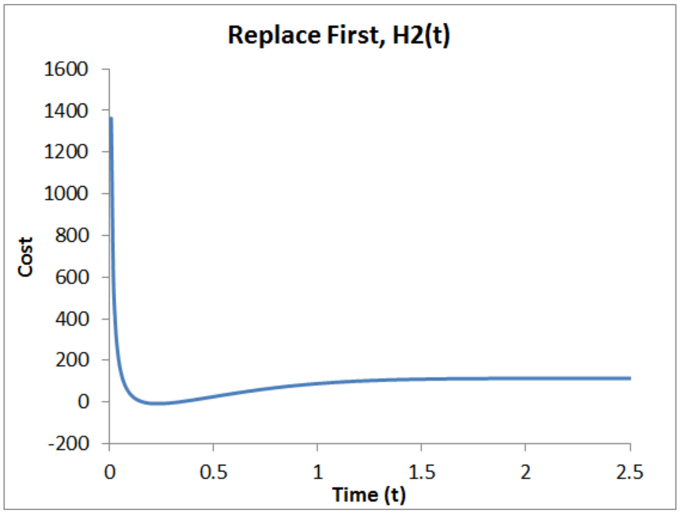

Figure 5. Replace first cost function $H_2(t)$

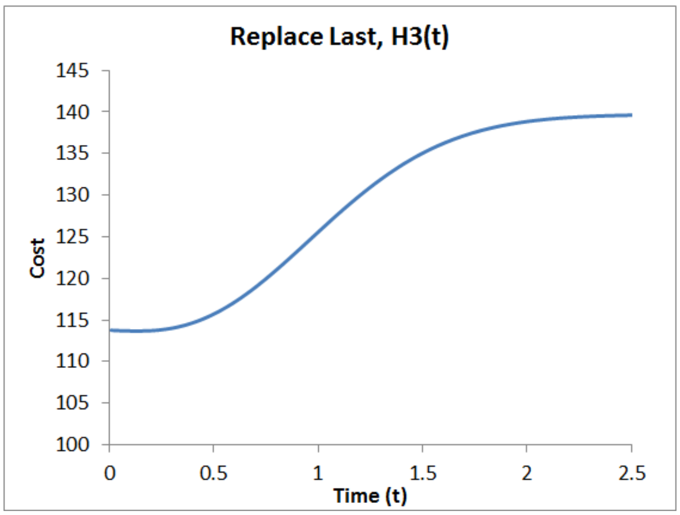

Figure 6. Replace Last cost function $H_3(t)$

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